

ac-driven Brownian motors: a Fokker-Planck treatment

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Abstract

We consider a primary model of ac-driven Brownian motors, i.e., a classical particle placed in a spatial-time periodic potential and coupled to a heat bath. The effects of fluctuations and dissipations are studied by a time-dependent Fokker-Planck equation. The approach allows us to map the original stochastic problem onto a system of ordinary linear algebraic equations. The solution of the system provides complete information about ratchet transport, avoiding such disadvantages of direct stochastic calculations as long transients and large statistical fluctuations. The Fokker-Planck approach to dynamical ratchets is instructive and opens the space for further generalizations.

I. INTRODUCTION

When one starts to pan for gold, he/she places a gold bearing sand (or gravel) into a pan and then agitates it in a circular motion, back and forth, left and right. Finally, the sand goes away with the water and gold grains remain in the bottom of the pan. However, if one simply inclines the pan then all the content will go away.

This is the illustration of the basic mechanism which lies behind the ratchet idea: in a driven system without a preferable direction of motion, transport velocities depend on characteristics of movable objects – charges, spins, masses, sizes, etc^{1,2,3,4}. The intensive development of this idea during the last decade brought new tools for a smart control of transport in different systems, ranging from mechanical engines⁵ to quantum devices⁶.

The ratchet setup demands three key ingredients which are (i) nonlinearity, (ii) asymmetry (spatial and/or temporal), and (iii) fluctuating input force of zero mean. The nonlinearity, in a sense of a nonlinear system response to an external force, is necessary since, otherwise, the system will produce a zero-mean output from a zero-mean input. The asymmetry is needed to violate the left/right symmetry of the response because transport in a fully symmetrical system is unbiased. The zero-mean fluctuating force should break thermodynamical equilibrium, which forbids appearance of a directed transport due to the Second Law of Thermodynamics^{7,8}.

The basic model of dynamical ratchets is an one-dimensional classical particle in a periodic potential exposed to an ac-field^{9,10,11,12}. While the symmetry analysis of microscopic equations of motion⁹ enables one to formulate necessary conditions for the directed transport appearance¹³, both the current sign and strength depend on dynamical mechanisms. At the deterministic limit, when noise is absent, the evolution of a damped particle is governed by attractors, regular (limit cycles) or chaotic ones¹⁴. Transport properties are encoded in the characteristics of attractors, so when there is only one attractor in phase space, the dc-current is equal to a mean velocity of the attractor¹².

The situation changes drastically when noise starts to contribute to the dynamics. During the evolution, particle jumps out of an attractor, evolves outside of the attractor vicinity, lands back into the attractor, etc. In other words, the particle explores the *whole phase space* and the dynamics of the particle can not be described in terms of the attractor properties only. The situation becomes even more complicated when several attractors coexist in phase

space¹⁵.

The standard approach, based on the direct Langevin simulations of the microscopic equations of motion¹⁶, demands a very long time in order to overcome transient effects and to produce the sufficient self-averaging over a phase space. The goal of this paper is to show that all these problems can be tackled by using the time-dependent Fokker-Planck equation (FPE)^{17,18}. Our approach allows us to reduce the stochastic problem to a system of ordinary linear algebraic equations, which can be solved by using standard numerical routines. The solution of the system of equations provides a full information on transport properties of the system.

The outline of this article is as follows. In Sec. II we introduce the model and set up the problem within the Fokker-Planck frame. Then, we formulate the symmetries which need to be violated in order to get a nonzero dc-current. Section III discusses the method of the solution of ac-driven Fokker-Planck equations. In Sec. IV, we illustrate our approach with the so-called tilting ratchets⁴. Sec. V contains some conclusions and perspectives.

II. THE MODEL

The one-dimensional classical dynamics of a particle (e.g. a cold atom placed into an optical lattice, or a colloidal microsphere in a magnetic bubble lattice³), of mass m and friction coefficient γ , exposed to an ac-driven periodic potential and coupled to a heat bath, is described by the equation:

$$m\ddot{x} + \gamma\dot{x} = g(x, t) + \xi(t), \quad g(x, t) = -\partial U(x, t)/\partial x, \quad (1)$$

where $U(x, t)$ is a potential and the force $g(x, t)$ is time and space periodic,

$$g(x + L, t) = g(x, t + T) = g(x, t). \quad (2)$$

The noise is modeled by a δ -correlated Gaussian white noise, $\langle \xi(t) \rangle = 0$, $\langle \xi(t)\xi(t') \rangle = 2\gamma D\delta(t - t')$, where the noise intensity is $D = kT$.

The state of the system (1) can be represented as a point in the three-dimensional phase space, (x, v, t) , for $m \neq 0$ (underdamped regime) and in the two-dimensional phase space, (x, t) , when $m = 0$ (overdamped limit)¹⁴. The stationary asymptotic current is equal to

$$J = \lim_{t \rightarrow \infty} J(t) = \lim_{t \rightarrow \infty} \frac{x(t)}{t}. \quad (3)$$

The statistical description of the system at $m = 1$ is provided by the the Fokker-Planck equation^{17,18}

$$\left\{ \partial_t + \frac{\partial}{\partial x} v - \frac{\partial}{\partial v} [\gamma v - g(x, t)] - \gamma D \frac{\partial^2}{\partial v^2} \right\} P(x, v, t) = 0, \quad (4)$$

where $v = \dot{x}$. The respective FPE for the overdamped limit, when inertia is negligible, $m = 0$, reads^{17,18}

$$\gamma \dot{P}(x, t) = - \left[\frac{\partial}{\partial x} g(x, t) - D \frac{\partial^2}{\partial x^2} \right] P(x, t). \quad (5)$$

The overdamped limit is the appropriate description for microdynamics at low Reynolds number, when a particle moves in an extremely viscous media¹⁹.

The Fokker-Planck equations, (4),(5), are linear, dissipative, and preserve the norm, $\int P dx dv$ for Eq. (4) and $\int P dx$ for Eq.(5)¹⁸. In addition, the equations possess discrete time and space translation symmetries, so that the operations $x \rightarrow x + L$ and $t \rightarrow t + T$ leave the equations invariant. For a given boundary condition and a fixed norm, any initial distribution, $P(\dots, 0)$, will converge to a single time-periodic attractor solution, $\tilde{P}(\dots, t) = \tilde{P}(\dots, t+T)$. What are the correct spatially boundary condition for the ratchet problem given by Eqs.(1-3)? It has been shown that the current can be calculated with the spatial-periodic solution of the type $\tilde{P}(x, \dots) = \tilde{P}(x + L, \dots)$ ⁴.

The dc-components of the directed current (3) in terms of the spatially-periodic attractor solution \tilde{P} are given by⁴:

$$J = \left\langle \int_{-\infty}^{\infty} v \cdot \hat{P}(x, v, t) \cdot dv \right\rangle_{T,L}, \quad m = 1, \quad (6)$$

$$J = \gamma^{-1} \langle g(x, t) \cdot \hat{P}(x, t) \rangle_{T,L}, \quad m = 0, \quad (7)$$

where $\langle \dots \rangle_T = \int_0^T \dots dt$ and $\langle \dots \rangle_L = \int_0^L \dots dx$. Without loss of generality, we set $L = 2\pi$.

Let us assume that Eq.(4) (or Eq.(5)) is invariant under some transformation of the variables x and t , which does not affect the boundary conditions. Then the unique solution \tilde{P} is also invariant under the transformation. The strategy is now to identify symmetry operations which invert the sign of the current J (6) (or (7)) and, at the same time, leave the corresponding FPE invariant. If such a transformation exists, the dc-current J will strictly vanish²⁰. Sign changes of the current can be obtained by either inverting the spatial coordinate x , or time t (plus simultaneous velocity inversion, $v \rightarrow -v$, for the underdamped case).

Below we list all such transformations together with the requirements the force $g(x, t)$ has to fulfill^{9,20}:

$$\widehat{S}_1 : x \rightarrow -x + x', \quad t \rightarrow t + t'; \quad \widehat{S}_1(g) \rightarrow -g, \quad (8)$$

$$\widehat{S}_2 : x \rightarrow x + x', \quad t \rightarrow -t + t'; \quad \widehat{S}_2(g) \rightarrow g, \quad (\text{if } \gamma = 0). \quad (9)$$

Here x' and t' depend on the particular shape of $g(x, t)$.

There is an additional symmetry for the overdamped limit²¹,

$$\widehat{S}_3 : x \rightarrow x + x', \quad t \rightarrow -t + t'; \quad \widehat{S}_3(g) \rightarrow -g, \quad (\text{if } m = 0). \quad (10)$$

which does not follow from the FPE (5). This is not a symmetry of the equation (due to the last r. h. s. term in Eq.(5), so-called diffusive term¹⁷). It has been shown, nevertheless, that the symmetry \widehat{S}_3 corresponds to a certain property of the asymptotic attractor solution $\tilde{P}(x, t)$, such that the current (7) exactly vanishes when \widehat{S}_3 holds²⁰.

By a proper choice of $g(x, t)$ all relevant symmetries can be broken, and then one can expect the appearance of a non-zero dc current J (6-7).

III. METHOD OF SOLUTION

It is impossible to solve the FPE's (4-5) analytically. Therefore we deal with the equations by using the Fourier expansion (over the x and t) for the overdamped case (5),

$$P(x, t) = \frac{1}{\sqrt{2\pi T}} \sum_{n,k=-N,-K}^{N,K} P_{nk} \cdot e^{i(nx+k\omega t)}, \quad (11)$$

plus the matrix continued fraction technique¹⁷ for the underdamped regime (4),

$$P(x, v, t) = \frac{\psi_0(v)}{\sqrt{2\pi T}} \sum_{n,k=-N,-K}^{N,K} \sum_{s=0}^S P_{nks} \cdot e^{i(nx+k\omega t)} \psi_s(v), \quad (12)$$

where $\psi_s(v)$ is the Hermite function of s -order²². Both the expansions (11-12) are truncated, so we can control the convergence and the precision of the solution \tilde{P} by variation of the parameters N , K and S .

By inserting the expansions (11-12) into the corresponding FPEs (5-4), we obtain the following systems of linear algebraic equations,

$$(ik\omega + Dn^2)P_{lk} + in \sum_{l,q} g_{lq} P_{(n-l)(k-q)} = 0, \quad (13)$$

for the overdamped limit, and

$$(ik\omega + s\gamma D)P_{nks} + i\sqrt{D}n(\sqrt{s}P_{nk(s-1)} + \sqrt{s+1}P_{nk(s+1)}) - \sqrt{\frac{s}{D}} \sum_{l,q} g_{lq}P_{(n-l)(k-q)(s-1)} = 0, \quad (14)$$

for the underdamped regime. Here g_{nk} is the Fourier element of the force function, $g(x, t) = \sum_{n,k} g_{nk} e^{i(nx+k\omega t)}$. We note that the equation for the dc-element, P_{00} in Eq.(5) (P_{000} in Eq.(4)), should be replaced by the equation $P_{00} = 1$ ($P_{000} = 1$). This corresponds to the normalization condition, $\int_0^{2\pi} P(x, t) dx = 1$ ($\int_0^{2\pi} \int_{-\infty}^{\infty} P(x, v, t) dx dv = 1$).

The definition for the current takes the form

$$J = \sqrt{\frac{D}{LT}} \cdot P_{001}, \quad (15)$$

for the underdamped case, and

$$J = \frac{1}{\sqrt{LT}} \sum_{n,k} g_{nk} \cdot P_{-n-k}, \quad (16)$$

for the overdamped limit.

In order to use the standard routine we have to transform the original variables, P_{nk} (overdamped limit) and P_{nks} (underdamped regime), into the single-index variable \mathcal{P}_z , $z = 1, \dots, Z$. There is the one-to-one transformation,

$$\{n, k\} \rightarrow z = 1 + (n + N)(2K + 1) + k + K \quad (17)$$

$$Z = (2N + 1)(2K + 1), \quad (18)$$

for the overdamped limit, and

$$\{n, k, s\} \rightarrow z = 1 + s(2N + 1)(2K + 1) + (n + N)(2K + 1) + k + K \quad (19)$$

$$Z = (2N + 1)(2K + 1)(S + 1), \quad (20)$$

for the underdamped regime, which transforms the original two- and three-dimensional matrices, P_{nk} and P_{nks} , into the vector-column \mathcal{P}_z .

The corresponding system of equations for \mathcal{P}_z can be solved by using the standard numerical routines. For our calculations, we have used the routine F07ANF from Linear Algebra Package (LAPACK) written in Fortran 77²³.

IV. EXAMPLE: TILTING RATCHETS

There is a plenty of choices for a driving potential $U(x, t)^4$. Below we consider the case of a particle placed onto a stationary periodic potential and driven by an alternating tilting force^{9,11,12,20},

$$g(x, t) = -V'(x) + E(t). \quad (21)$$

With the simple potential $V(x) = V_0(1 - \cos(x))$, the two-frequency driving,

$$E(t) = E_1 \sin(\omega t) + E_2 \sin(2\omega t + \theta), \quad (22)$$

ensures that for $E_1, E_2 \neq 0$ \hat{S}_1 is always violated. The symmetry \hat{S}_2 is broken for $\theta \neq 0, \pm\pi$. In addition, \hat{S}_3 does not hold at $\theta \neq \pm\pi/2$ ²⁰.

We start from the overdamped limit, $m = 0$ and $\gamma = 1$. For a given set of parameters, $V_0 = 2$, $E_1 = 4.6$, $E_2 = -6$, and $\omega = 0.75$, there is only one limit cycle in the phase space of the system (inset in Fig.1). The limit cycle is nontransporting, i. e. particle returns to the initial position after one period of external driving T , for the whole range $\theta \in [-\pi, \pi]$. Therefore, there is no dc-transport in the deterministic limit, $D = 0$.

When noise enters the game, it changes the dynamics: particle can leave the attractor and this leads to a finite current appearance (Fig.1). Therefore, thermal fluctuations play a constructive role here by providing a way for the manifestation of the asymmetry hidden in the ac-driving. Since the current depends nonmonotonously on the noise intensity D , there is some kind of the stochastic resonance effect^{24,25}. For a weak noise, the dynamics is still localized near the attractor (Fig. 1a, dashed line), so that the current is faint. In the opposite high-temperature limit, when the noise starts to dominate the dynamics, the particle dynamics is "smeared" over the phase space. The relevant space-temporal correlations are suppressed by the noise and the current tends to decrease. There is a resonance temperature (near $D \approx 0.27$, Fig.1b), where the dc-current can be resonantly enhanced. Here the directed flow of particles is analogous to the flow of information in the case of stochastic resonance^{24,25}.

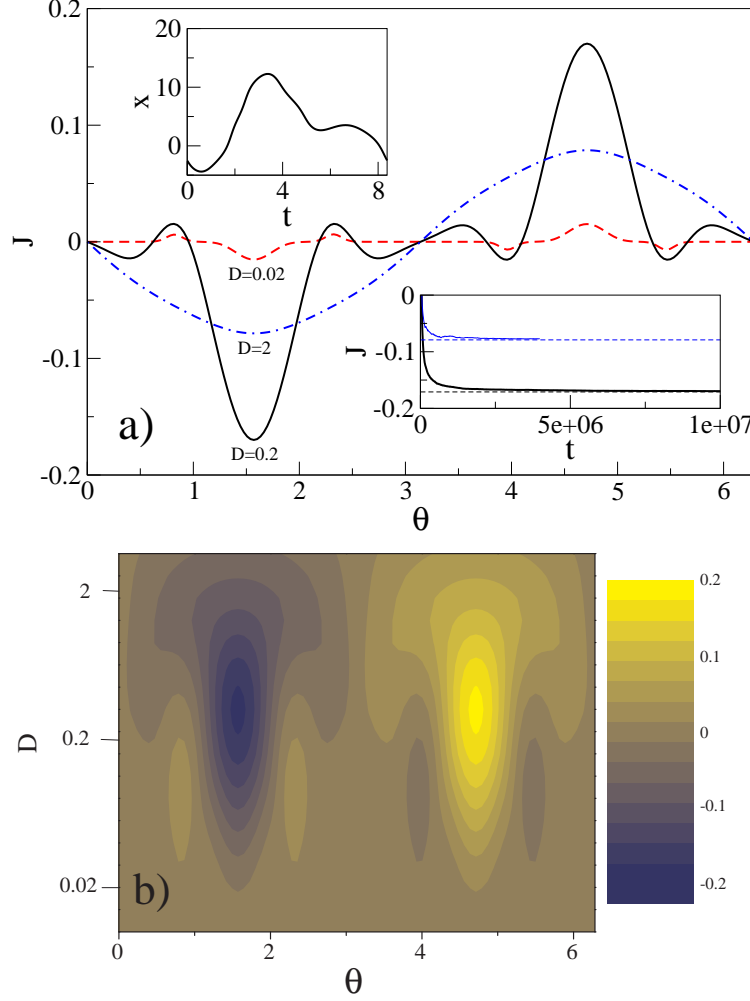


FIG. 1: (color online)(a) Dependence of the current J (16) on θ , for the system (13) ($N = K = 60$), at different temperatures, $D = kT$. The parameters are $\gamma = 1$, $V_0 = 2$, $E_1 = 4.6$, $E_2 = -6$, and $\omega = 0.75$. Insets: the attractor of the corresponding deterministic system (1) for $\theta = -\pi/2$ (top left); the dependence of the running current $J(t) = x(t)/t$ on t at the point $\theta = -\pi/2$, for $D = 0.2$ (thick line) and $D = 2$ (dashed dotted line). Dashed lines correspond to the current values J (16) (bottom right); (b) More detailed dependence: the current J (16) as a function of θ and D .

Note, that the noise variation can not violate the basic symmetries: current is always absent at the point $\theta = 0$ and $\theta = \pm\pi$. We have corroborated our analysis by the direct Langevin simulation (inset on Fig. 1) and found a perfect agreement with the numerical solutions of the corresponding FPE.

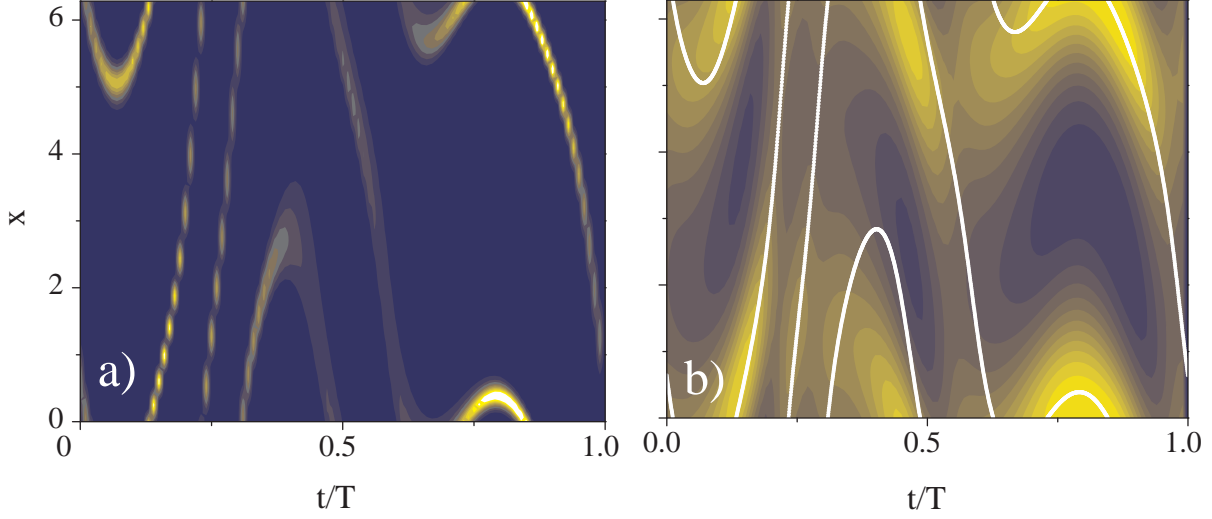


FIG. 2: (color online) The attractor solution $\tilde{P}(x, t)$ of the FPE (5), for $D = 0.02$ (a) and $D = 2$ (b). The white line in panel (b) marks the corresponding deterministic attractor. Here $\theta = -\pi/2$, while the other parameters are the same as in Fig.1.

The overdamped limit, however, is not suitable for the description of all realistic situations. For example, for the modeling of Josephson ratchets it is necessary to take into account the inertia term, $m\ddot{x}$ ²⁶. The underdamped regime without periodic driving has been studied for a long time. To the best of our knowledge, there is only one paper where FPE for an underdamped system with ac-driving has been solved numerically²⁷.

The results for $m = 1$, $V_0 = 1$, $E_1 = E_2 = 1.2$, $\omega = 1$, $\gamma = 0.1$, and $\theta = 0$ are shown in Fig.3. There is a single chaotic attractor for the deterministic limit, $D = 0$. In Fig.3a the Poincare section of the attractor (the position of the particle in the phase space, $[x(t), v(t)]$, taken at stroboscical times $t_n = nT^{14}$), is depicted. Since the relevant symmetry, S_1 , is violated by the presence of the second harmonics E_2 , the attractor generates a finite dc-current, $J \approx -0.0288$ (Fig.4).

Although the attractor solution of the corresponding FPE for $D = 0.1$ clearly resembles its deterministic counterpart (Fig.3), it produces a much stronger current, $J \approx -0.083$, than in the deterministic limit. Here, again, thermal fluctuations play a constructive role in the process of dc-current rectification.

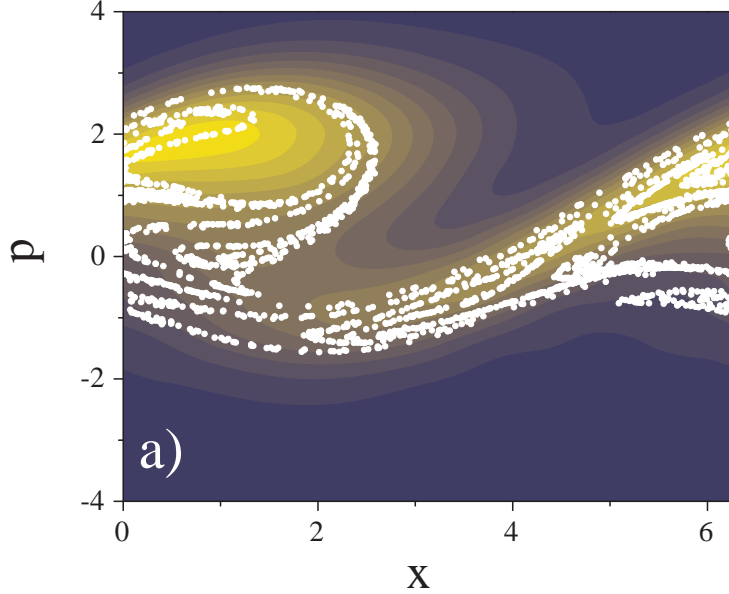


FIG. 3: (color online) The stroboscopic representation of the attractor solution $\tilde{P}(x, v, 0)$ (12) of the Eq. (14) ($N = K = 35, S = 25$), with the Poincare section of the corresponding deterministic attractor (1) (white dots). The parameters are $m = 1$, $V_0 = 1$, $E_1 = E_2 = 1.2$, $\omega = 1$, $\gamma = 0.1$, and $\theta = 0$.

The convergence of the quantity (15) with increasing of the total number of basic states, Z (20), is relatively slow (Fig.4). We have to take into account 71 spatial and 71 temporal harmonics in order to estimate accurately the ratchet current. This fact means that the fast temporal modes (chaotic dynamics on a short time scale, $t_c \ll T$) and short spatial modes ($x_c \ll L$), contribute essentially to the overall ratchet current.

V. CONCLUSIONS

We have shown that the complex issue of ac-driven ratchets in a thermal environment can be resolved by using the Fokker-Planck equation. Our approach maps the original problem onto a set of ordinary algebraic linear equations, which then can be solved by a standard numerical routine.

The approach opens a new perspectives for further developments. For example, it allows a generalization for the case of colored noise with exponential correlation function¹⁷. The corresponding one-dimensional "non-thermal" process can be embedded into a two-dimensional

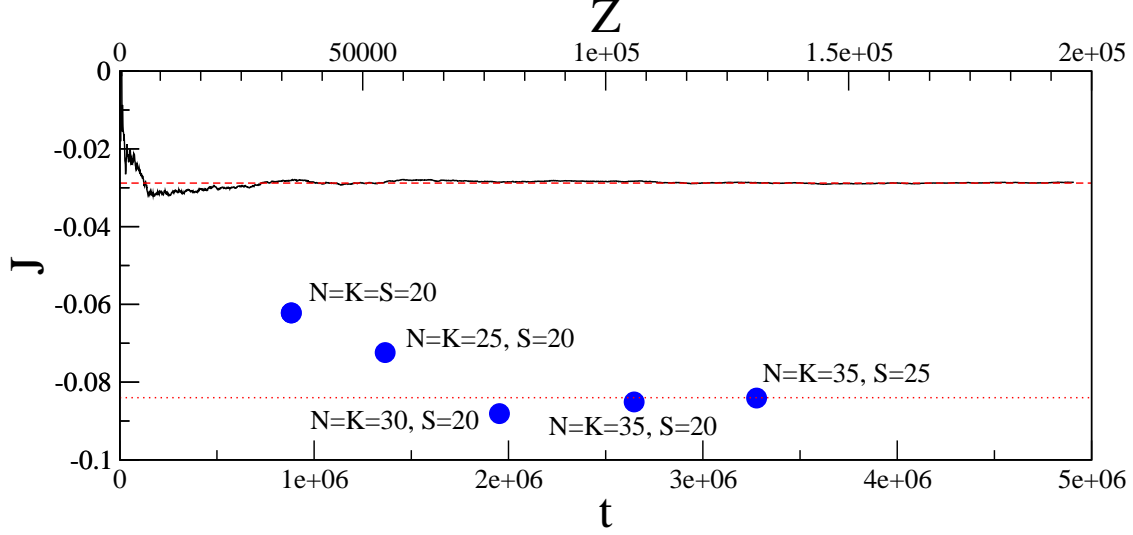


FIG. 4: (color online) The running current $J(t) = x(t)/t$ vs t for deterministic system (1) and the dependence of the current J (15) on the number of basic states Z (20). The other parameters are the same as in Fig.3.

”thermal” dynamics by introducing the additional dynamical variable, $\varepsilon(t)$, which should replace the noise term, $\xi(t)$, in Eq.(1)²⁸. The next prominent perspective is the quantum limit of underdamped ac-driving ratchets. The master equation for the corresponding Wigner function can be represented as a FPE (14) with an additional coupling between elements²⁹. Finally, the absolute negative mobility in ac-driven inertial systems^{30,31} is another possible target for our method.

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